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# Detecting Jumps in High-Frequency Financial Series Using Multipower Variation\*

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## Abstract

When the log-price process incorporates a jump component, realized variance will no longer estimate the integrated variance since its probability limit will be determined by the continuous and jump components. Instead realized bipower variation, tripower variation and quadpower variation are consistent estimators of integrated variance even in the presence of jumps. In this paper we derive the limit distributions of realized tripower and quadpower variation, allowing us to compare these three estimators of integrated variance. Using the limit theories for the differences of the errors, tests for jumps are proposed for each estimator. Using simulated data, the performance of each of these tests is compared. The tests are also applied to empirical data but results need to be taken carefully as market microstructure effects may contaminate real data.

**Keywords:** Quadratic variation, Multipower variation, Stochastic volatility models, Jump process, Semimartingale, High-frequency data.

**JEL Classification:** C12, C51, G19.

## Resumen

Cuando el proceso del logaritmo de precios incorpora un componente de brincos, la varianza realizada ya no estimará la varianza integrada dado que su límite en probabilidad será determinado tanto por el componente continuo como por el de brincos. En cambio, la variación bipoder realizada, la variación tripoder realizada y la variación cuadripoder realizada son estimadores consistentes de la varianza integrada aún en la presencia de brincos. En este artículo se derivan las distribuciones límites de la variación tripoder y la variación cuadripoder permitiendo comparar estos tres estimadores de la varianza integrada. Usando las teorías asintóticas de las diferencias de los errores, pruebas de hipótesis para detectar brincos en el proceso se propone para cada estimador. Usando series simuladas, se comparan dichas pruebas. Éstas también son aplicadas a datos empíricos pero los resultados deben ser tomados con cuidado dado que las series reales suelen estar contaminadas por efectos de microestructura.

**Palabras Clave:** Variación cuadrática, Variación multipoder, Modelos de volatilidad estocástica, Proceso de brinco, Semimartingala, Datos en alta frecuencia.

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# 1. Introduction

Analysing properties of financial structures is a major focus of financial statistics. The issues at stake are the description of price dynamics and the development of concepts and theories to this effect. Markets have become increasingly volatile and new tools are required to quantify the magnitude of fluctuations. Today, complete records of prices are available for many financial assets at a high-frequency and this enables us to calibrate continuous time models. More precisely, within a continuous semimartingale process realised variance estimates quadratic variation. It can be used as an estimator of integrated variance in a stochastic volatility model when high-frequency data is available.

Even though realised variance consistently estimates integrated variance under a continuous log-price process, whenever log-prices incur jumps, realised variance will estimate the quadratic variations of both continuous and jump components. Jumps are believed to be a fundamental component of financial processes, therefore the need of estimators robust to the presence of jumps. Recently, econometric methods have been developed to split the continuous and jump components using bipower variation (Barndorff-Nielsen and Shephard(2004a)). In this paper we will consider alternative consistent estimators of integrated variance in the presence of finite activity jumps. Such estimators are the tripower and quadpower variation, special cases of multipower variation. These objects are known to have better sample behaviour than bipower variation when estimating integrated quarticity, consequently we may also expect a better sample behaviour when estimating integrated variance.

In the case of a stochastic volatility process combined with jumps, the difference between realised variance and these realised objects gives an estimation of the quadratic variation of the jump component. These properties are at the basis of a test for jumps in the log-price sample path.

To this effect we will derive the asymptotic distribution of these estimators and later, we will use these asymptotic results to test for jumps in the log-price process. The power of our estimators can then be compared to those of bipower variation found in studies by Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2006), Huang and Tauchen (2005), Tauchen and Zhou (2006).

Realised variance and multipower variation may suffer from a bias problem due to autocorrelation in the intra-day returns. It has many sources referred to as market microstructure effects. These effects induce serial correlation in high-frequency returns, used to calculate realised variance or multipower variation; therefore they have an impact on the integrated variance estimation. Nevertheless, market microstructure noise will not be incorporated to our log-price process in this paper. We think that multipower variation may be more robust to market microstructure effects but further research is needed to assess the impact of this noise on the estimators and tests.

The outline of this paper is as follows. Section 2 reviews some definitions and results for quadratic and bipower variation with corresponding results for tripower and quadpower variation given in Section 3. In Section 4 we present the asymptotic theory of these estimators and in Sections 5 and 6 we test for jumps on simulated and empirical data respectively. Section 7 concludes. The proofs of the asymptotic theories can be found in the Appendix.

## 2. Quadratic and bipower variation

### 2.1. Quadratic variation

A standard model in financial economics is a stochastic volatility (SV) model for log-prices  $Y_t$  which follows the equation

$$Y_t = \int_0^t a_u du + \int_0^t \sigma_s dW_s, \quad t \geq 0, \quad (1)$$

where  $A_t = \int_0^t a_u du$ . The processes  $\sigma_t$  and  $A_t$  are assumed to be stochastically independent of the standard Brownian motion  $W$ . Here  $\sigma_t$  is called the instantaneous or spot volatility,  $\sigma_t^2$  the corresponding spot variance and  $A_t$  the mean process.

More generally  $A_t$  is assumed to have locally bounded variation paths and it is set that  $M_t = \int_0^t \sigma_s dW_s$ , with the added condition that  $\int_0^t \sigma_s^2 ds < \infty$  for all  $t$ . This is enough to guarantee that  $M_t$  is a local martingale. So the original equation (1) can be decomposed as

$$Y_t = A_t + M_t.$$

Under these assumptions  $Y_t$  is a semimartingale (see Protter(1990)). If additionally  $A_t$  is continuous then  $Y_t$  is a member of the continuous stochastic volatility semimartingale ( $SVSM^c$ ) class.

Here, a key role is played by the integrated variance

$$\sigma_t^{2*} = \int_0^t \sigma_s^2 ds,$$

and the quadratic variation

$$[Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2$$

for any sequence of partitions  $t_0^{(n)} = 0 < t_1^{(n)} < \dots < t_n^{(n)} = t$  with  $\sup_j \{t_j^{(n)} - t_{j-1}^{(n)}\} \rightarrow 0$  for  $n \rightarrow \infty$ . As  $A_t$  is assumed to be continuous and of finite variation we obtain that

$$[Y]_t = [A]_t + 2[A, M]_t + [M]_t = [M]_t = \int_0^t \sigma_u^2 du$$

where

$$[X, Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}}).$$

This holds since the quadratic variation of any continuous, locally bounded variation process is zero (see Hull and White (1987)).

Assume that we have observations every  $\delta > 0$  periods of time, so given the previous framework, let the  $j$ -th return be

$$y_j = Y_{j\delta} - Y_{(j-1)\delta} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor,$$

and the actual variance be

$$\nu_j^2 = \sigma_{j\delta}^{2*} - \sigma_{(j-1)\delta}^{2*}.$$

If high-frequency financial data is available, the realised variance process

$$[Y_\delta]_t^{[2]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

can be defined.

The relationship between realised variance and quadratic variation is well known to be

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t = \int_0^t \sigma_s^2 ds$$

if  $Y \in SVSM^c$ .

In Barndorff-Nielsen and Shephard (2002), Barndorff-Nielsen and Shephard (2003) and Barndorff-Nielsen and Shephard (2004b) the previous theory has been extended to a Central Limit Theorem (CLT). In these papers the CLT is presented under somewhat restrictive assumptions. Recently Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) give weaker conditions on the log-price process which ensure that the CLT holds. So for the SV model (1), when  $\delta \downarrow 0$

$$\frac{\delta^{-1/2}([Y_\delta]_t^{[2]} - [Y]_t)}{\sqrt{2 \int_0^t \sigma_s^4 ds}} \xrightarrow{L} N(0, 1),$$

under the assumptions that  $A_t$  is of locally bounded variation,  $\int_0^t \sigma_u^2 du < \infty$  and that  $\sigma_t$  is càdlàg.

Realised variance has been used in financial econometrics for many years, examples include Rosenberg (1972), Merton (1980), Poterba and Summers (1986), Schwert (1989), Hsieh (1991), Zhou (1996), Taylor and Xu (1997), Christensen and Prabhala (1998), Andersen, Bollerslev, Diebold and Ebens (2001), Andersen, Bollerslev, Diebold and Labys (2001). Recent literature using quadratic variation for semimartingales has been the independently and concurrently development of Andersen and Bollerslev (1998), Comte and Renault (1998) and Barndorff-Nielsen and Shephard (2001). Many other papers on realised variance exist which are discussed in Dacorogna et al. (2001) and in the reviews by Andersen, Bollerslev and Diebold (2005) and Barndorff-Nielsen and Shephard (2007).

## 2.2. Bipower variation

Barndorff-Nielsen and Shephard (2004a) introduced the bipower variation process, defined as

$$\{Y\}_t^{[r,s]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j|^r |y_{j+1}|^s, \quad r, s \geq 0.$$

Our main interest is the estimation of integrated variance, therefore it is necessary to set  $r + s = 2$ . Here we will focus on the case where  $r = s = 1$  as limit theorems can be obtained under rather weak conditions. Then the realised bipower variation process

$$\{Y_\delta\}_t^{[1,1]} = \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j| |y_{j+1}|$$

is an estimator of  $\{Y\}_t^{[1,1]}$ . These authors show that in the  $SVSM^c$  case under some regularity conditions

$$\mu_1^{-2}\{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

Furthermore, in the  $SVSM^c$

$$[Y_\delta]_t^{[2]} - \mu_1^{-2}\{Y_\delta\}_t^{[1,1]} \xrightarrow{p} 0.$$

### 2.3. Effect of jumps

We now consider the following process consisting of a continuous and a jump component

$$Y_t = Y_t^{(1)} + Y_t^{(2)}$$

where  $Y^{(1)} \in SVSM^c$  and

$$Y_t^{(2)} = \sum_{i=1}^{N_t} c_i$$

with  $N_t < \infty$  for all  $t > 0$  a finite activity simple counting process and  $\{c_i\}$  a collection of non-zero random variables.

Notice that in this case the quadratic variation is

$$\begin{aligned} [Y]_t &= \sigma_t^{2*} + \sum_{i=1}^{N_t} c_i^2 \\ &= [Y^{(1)}]_t + [Y^{(2)}]_t, \end{aligned}$$

so both components, the continuous and the jump one, contribute to the total quadratic variation.

In Barndorff-Nielsen and Shephard (2004a) it is shown that the realised variance incorporates the contribution to the quadratic variation of both the continuous and jump components

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t$$

but the realised bipower variation just incorporates the contribution of the continuous one

$$\mu_1^{-2}\{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

These results imply that in this case

$$[Y_\delta]_t^{[2]} - \mu_1^{-2}\{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2.$$

### 2.4. Testing for jumps using Realised bipower variation

By using realised variance and realised bipower variation, it is possible to identify the continuous component and the jump component of the log-price process, so to test for jumps in the log-price process we only need to know the convergence in distribution of the estimators. In Barndorff-Nielsen and Shephard (2004a) the joint distribution theory for the realised variance error and the realised bipower variation error was firstly given. Recently Barndorff-Nielsen, Graversen, Jacod, Podolskij

and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) have unified the asymptotic treatment of some volatility measures, where the following result can be seen as a special case. The necessary limit theory for the difference of the realised variance and the realised bipower variation is

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( [Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \right) \xrightarrow{L} N(0, \vartheta_{RV})$$

where  $\vartheta_{RV} \simeq 0.6091$ . This theory is later used by Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2006) and Huang and Tauchen (2005) to carry out the testing.

Here we assume that the jump component of the log-prices is a finite activity jump process, though Barndorff-Nielsen, Shephard and Winkel (2006) obtained the conditions for the convergence in probability and central limit theorem to hold under an infinite activity jump process.

### 3. Tripower and Quadpower variation

Barndorff-Nielsen and Shephard (2004a) generalised bipower variation by multiplying together a finite number of absolute returns raised to some non-negative power, calling this multipower variation. Tripower variation and quadpower variation are particular cases of this idea. The tripower variation process is defined as

$$\{Y\}_t^{[r,s,u]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s+u)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u, \quad r, s, u > 0,$$

and estimated with the realised tripower variation process

$$\{Y_\delta\}_t^{[r,s,u]} = \delta^{1-(r+s+u)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u, \quad r, s, u > 0.$$

Here we need that  $r + s + u = 2$ , so we will focus on the case where  $r = s = u = 2/3$  to be able to obtain the limit distributions under weak assumptions. Analogously the quadpower variation process is defined as

$$\{Y\}_t^{[r,s,u,v]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s+u+v)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u |y_{j+3}|^v, \quad r, s, u, v > 0$$

and estimated with the realised quadpower variation process

$$\{Y_\delta\}_t^{[r,s,u,v]} = \delta^{1-(r+s+u+v)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u |y_{j+3}|^v, \quad r, s, u, v > 0.$$

Again  $r + s + u + v = 2$  and we will consider the special case where  $r = s = u = v = 1/2$ .

Given that  $Y \in SVSM^c$ , we can get new estimators of the integrated variance based on the following results

$$\mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \xrightarrow{p} \int_0^t \sigma_s^2 ds,$$

and

$$\mu_{1/2}^{-4}\{Y_\delta\}_t^{[1/2,1/2,1/2,1/2]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

So we can also obtain that

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3}\{Y_\delta\}_t^{[2/3,2/3,2/3]} \xrightarrow{p} 0$$

and that

$$[Y_\delta]_t^{[2]} - \mu_1^{-4}\{Y_\delta\}_t^{[1/2,1/2,1/2,1/2]} \xrightarrow{p} 0.$$

### 3.1. Effect of jumps

When a jump component is present in the log-price process, as in subSection 2.3, tripower variation and quadpower variation does not incorporate the contribution to quadratic variation of the jump component, so

$$\mu_{2/3}^{-3}\{Y_\delta\}_t^{[2/3,2/3,2/3]} \xrightarrow{p} \int_0^t \sigma_s^2 ds$$

and

$$\mu_{1/2}^{-4}\{Y_\delta\}_t^{[1/2,1/2,1/2,1/2]} \xrightarrow{p} \int_0^t \sigma_s^2 ds$$

These results imply that

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3}\{Y_\delta\}_t^{[2/3,2/3,2/3]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2$$

and that

$$[Y_\delta]_t^{[2]} - \mu_{1/2}^{-4}\{Y_\delta\}_t^{[1/2,1/2,1/2,1/2]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2.$$

As a consequence of this, tripower and quadpower variation provide an alternative to bipower variation in testing for jumps, we only need the asymptotic distributions which will be obtained in the next section.

Barndorff-Nielsen, Shephard and Winkel (2006) generalised these results, showing that multipower variation,  $d_r\{Y_\delta\}_t^{\{r_1, \dots, r_s\}}$  with  $\sum_{i=1}^s r_i = 2$ , converges in probability to integrated variance, when a jump process of finite activity is added to the log-price process, if  $\max\{r_1, \dots, r_s\} < 2$ . Here  $d_r$  is a constant that only depends on  $r_1, \dots, r_s$ .

### 3.2. Daily time series

To produce daily time series, assume a fixed time interval  $h > 0$  (here it denotes the period of a day) with  $\lfloor t/\delta \rfloor = M$  intra- $h$  returns, during each fixed  $h$  time period, defined as

$$y_{j,i} = Y_{(i-1)h+j\delta} - Y_{(i-1)h+(j-1)\delta},$$

for the  $j - th$  intra- $h$  return in the  $i - th$  period.

The daily realised tripower variation (RTV), can then be defined as

$$\mu_{2/3}^{-3}\{Y_M\}_i^{[2/3,2/3,2/3]} = \mu_{2/3}^{-3} \sum_{j=1}^{M-2} |y_{j,i}|^{2/3} |y_{j+1,i}|^{2/3} |y_{j+2,i}|^{2/3}$$



and the daily realised quadpower variation (RQV) as

$$\mu_{1/2}^{-4} \{Y_M\}_i^{[1/2, 1/2, 1/2, 1/2]} = \mu_{1/2}^{-4} \sum_{j=1}^{M-3} |y_{j,i}|^{1/2} |y_{j+1,i}|^{1/2} |y_{j+2,i}|^{1/2} |y_{j+3,i}|^{1/2}.$$

We will also study the skipped version of the daily realised bipower variation (RSBV)

$$\mu_1^{-2} \{Y_M\}_i^{[1, 0, 1]} = \sum_{j=1}^{M-2} |y_{j,i}| |y_{j+2,i}|$$

These three estimators converge in probability to the actual variance whether or not jumps are present in the log-price process. Furthermore, if such jumps are present their contribution to the quadratic variation,  $\sum_{j=N_{i-1}+1}^{N_i} c_j^2$ , can be estimated by the differences RV-RTV, RV-RQV and RV-RSBV.

## 4. Asymptotic theory

### 4.1. Asymptotic distributions for the difference between realised tripower, quadpower and skipped version of bipower variation and realised variance

To allow us to implement the test for jumps we need to extend from the convergence in probability to convergence in distribution. The asymptotic theory presented in this section and derived in the appendix is the main contribution of this paper. Later in this paper we will apply it for testing for jumps.

As in Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) let us assume that given the SV model in equation (1)  $A_t$  is of locally bounded variation and  $\sigma_t$  is càdlàg. These and additional more technical conditions on the driving process of  $\sigma_t$ , also stated in the previous papers, enable us to obtain the asymptotic distributions below. Additionally, here we will set  $A_t = 0$ .

#### Theorem 1

If  $Y \in SVSM^c$  then as  $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{TV})$$

where

$$\vartheta_{TV} = \mu_{4/3} \mu_{2/3}^{-2} (\mu_{4/3}^2 \mu_{2/3}^{-4} + 2\mu_{4/3} \mu_{2/3}^{-2} - 2) - 7 \simeq 1.0613$$

and  $\mu_r = E(|x|^r) = 2^{r/2} \frac{\Gamma(\frac{1}{2}(r+1))}{\Gamma(\frac{1}{2})}$ .

**Proof.** See appendix A.1.

**Theorem 2**

If  $Y \in SVSM^c$  then as  $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{QV})$$

where

$$\vartheta_{QV} = \mu_1 \mu_{1/2}^{-2} (\mu_1^3 \mu_{1/2}^{-6} + 2\mu_1^2 \mu_{1/2}^{-4} + 2\mu_1 \mu_{1/2}^{-2} - 2) - 9 \simeq 1.37702$$

and  $\mu_r = E(|x|^r)$ .

**Proof.** See appendix A.2.

**Theorem 3**

If  $Y \in SVSM^c$  then as  $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( \mu_1^{-2} \{Y_\delta\}_t^{[1, 0, 1]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{SBV})$$

where

$$\vartheta_{SBV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.60907$$

and  $\mu_r = E(|x|^r)$ .

**Proof.** See appendix A.3.

A consequence of this last limit theorem is that asymptotically there is no difference in distribution between working with the realised bipower variation or working with the realised skipped version of bipower variation.

From the previous asymptotic distribution we can observe that

$$\vartheta_{BV} = \vartheta_{SBV} < \vartheta_{TV} < \vartheta_{QV};$$

i.e. realised bipower variation is more efficient than realised tripower and quadpower variation. So in ideal conditions, when  $\delta \downarrow 0$  and there is no microstructure effect, we can expect that realised bipower variation will give us better results. Nevertheless, real financial series are finite and microstructure effect exists so it is uncertain which will give us the highest power in the test for jumps when working with empirical data.

Now that the three necessary asymptotic distributions have been obtained, the tests for jumps in the log-price process can be carried out.

## 4.2. Testing for jumps

Following Barndorff-Nielsen and Shephard (2006) two different tests for jumps can be established, a linear test and a ratio test. In both tests, under the null hypothesis, the price process does not include a jump component and under the alternative hypothesis, the price process consists of a continuous and a jump component. The jump component is assumed to be a finite activity jump process and the continuous component a member of the continuous stochastic volatility semimartingale class.

The linear tests are based on the previous limit theories (Theorem 1, 2 and 3) while the ratio tests utilize the following convergence results. For the realised tripower variation process

$$\frac{\left(\frac{\mu_{2/3}^{-3}\{Y_\delta\}_t^{[2/3,2/3,2/3]}}{[Y_\delta]_t^{[2]}} - 1\right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds\right)^2}}} \xrightarrow{L} N(0, \vartheta_{TV})$$

where  $\vartheta_{TV} \simeq 1.0613$ ; for the realised quadpower variation process

$$\frac{\left(\frac{\mu_{1/2}^{-4}\{Y_\delta\}_t^{[1/2,1/2,1/2,1/2]}}{[Y_\delta]_t^{[2]}} - 1\right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds\right)^2}}} \xrightarrow{L} N(0, \vartheta_{QV})$$

where  $\vartheta_{QV} \simeq 1.37702$ ; and for the realised skipped bipower variation process

$$\frac{\left(\frac{\mu_1^{-2}\{Y_\delta\}_t^{[1,0,1]}}{[Y_\delta]_t^{[2]}} - 1\right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds\right)^2}}} \xrightarrow{L} N(0, \vartheta_{SBV})$$

where  $\vartheta_{SBV} \simeq 0.60907$ .

When working with real data both the linear and ratio test are infeasible as  $\int_0^t \sigma_s^4 ds$  cannot be observed even when high-frequency data is available. In such situations the following estimators are needed

1) Realised Quarticity (E1)

$$M\mu_4^{-1}\{Y_M\}_i^{[4]} = M\mu_4^{-1} \sum_{j=1}^M y_{j,i}^4$$

2) Realised Tripower Variation with  $r = s = u = 4/3$  (E2)

$$M\mu_{4/3}^{-3}\{Y_M\}_i^{[4/3,4/3,4/3]} = M\mu_{4/3}^{-3} \sum_{j=1}^{M-2} |y_{j,i}|^{4/3} |y_{j+1,i}|^{4/3} |y_{j+2,i}|^{4/3}$$

3) Realised Quadpower Variation with  $r = s = u = v = 1$  (E3)

$$M\mu_1^{-4}\{Y_M\}_i^{[1,1,1,1]} = M\mu_1^{-4} \sum_{j=1}^{M-3} |y_{j,i}| |y_{j+1,i}| |y_{j+2,i}| |y_{j+3,i}|$$

Observe that the quadratic variation for the jump component cannot be negative, so the following estimators can be used

$$\begin{aligned} & \min(0, \mu_{2/3}^{-3} \{Y_M\}_i^{[2/3, 2/3, 2/3]} - [Y_M]_i^{[2]}), \\ & \min(0, \mu_{1/2}^{-4} \{Y_M\}_i^{[1/2, 1/2, 1/2, 1/2]} - [Y_M]_i^{[2]}), \\ & \min(0, \mu_1^{-2} \{Y_M\}_i^{[1, 0, 1]} - [Y_M]_i^{[2]}). \end{aligned}$$

As a result, we have one-sided tests.

A bias is introduced to the test statistics because we use finite values of  $M$ , therefore our estimators will have less components in the summation compared to the realised variance. This problem can be dealt with the modified estimators

$$\begin{aligned} & \left(\frac{M}{M-2}\right) \mu_{2/3}^{-3} \{Y_M\}_i^{[2/3, 2/3, 2/3]}, \\ & \left(\frac{M}{M-3}\right) \mu_{1/2}^{-4} \{Y_M\}_i^{[1/2, 1/2, 1/2, 1/2]}, \\ & \left(\frac{M}{M-2}\right) \mu_1^{-2} \{Y_M\}_i^{[1, 0, 1]}. \end{aligned}$$

In small samples in the feasible test, the result from Barndorff-Nielsen and Shepard(2003) given by Jensen's inequality,

$$\frac{\int_0^t \sigma_s^4}{(\int_0^t \sigma_s^2)^2} \geq 1$$

may not hold, so additionally we will use the adjusted ratio

$$\max\left(1, \frac{\widehat{\int_0^t \sigma_s^4 ds}}{(\widehat{\int_0^t \sigma_s^2})^2}\right).$$

## 5. Simulations

The test for jumps will be applied to a simple simulated process. A constant elasticity of variance (CEV) process will be used. Specifically the Feller (1951) or Cox, Ingersoll and Ross (1985) square root process for  $\sigma^2$

$$d\sigma_t^2 = -\lambda\{\sigma_t^2 - \xi\}dt + \omega\sigma_t dB_{\lambda t}, \quad \xi \geq \omega^2/2, \quad \lambda > 0,$$

where  $B$  is a standard Brownian motion process. The square root process has a marginal distribution

$$\sigma_t^2 \sim \Gamma(2\omega^{-2}\xi, 2\omega^{-2}) = \Gamma(\nu, a), \quad \nu \geq 1,$$

with a mean of  $\xi = \nu/a$  and a variance of  $\omega^2 = \nu/a^2$ . We will take  $A_t = 0$  and rule out the leverage effect by assuming  $Cor\{B_{\lambda t}, W_t\} = 0$ . We will take  $h = 1$ ,  $\lambda = 0.01$ ,  $\nu = 4$  and  $a = 8$ . The jumps will be i.i.d.  $N(0, \beta\nu/a)$ , thus a jump has the same variance as that expected over a  $(\beta \times 100)\%$  of a day when there are no jumps.

## 5.1. Finite sample behaviour of the estimators

### 5.1.1. Finite sample behaviour when there are no jumps

Firstly we will try to assess, given that there are no jumps, the accuracy of the convergence in probability for each of our estimators. To do so, we will simulate data corresponding to 1000 days from the previous process without jumps. The realised values for bipower, tripower, quadpower and the skipped version of bipower variation are calculated for different values of  $M$  ( $M=12, 72$  and  $288$ , i.e. 120, 20 and 5 minutes returns). Figures 1, 2, 3 and 4 correspond to the first 250 observations of realised bipower, tripower, quadpower and the skipped version of bipower variation respectively. The left hand graphs show the actual variance together with the realised values to assess the probability limit. As market microstructure noise is not added to our series, they become more accurate as the value of  $M$  increases. The magnitude of the difference seems to be large in times of high volatility. From these graphs, no particular estimator seems to outperform any of the others. The plots in the middle show the differences between the realised series and the actual variance, i.e. the errors, together with their confidence interval. The drastic fluctuations in the bands correspond to changes in the level of the variance process, with wider bands at levels of higher variance. The right hand graphs give the corresponding QQ-plot which should be a 45 degree line if the asymptotic results hold. There is an improvement in all the cases as  $M$  increases although the asymptotic theory does not provide an accurate guide to the finite sample behaviour. Even when using 5 minutes returns, there are problems due to heavy tails. Overall nothing can be concluded about which estimator shows better finite sample behaviour.

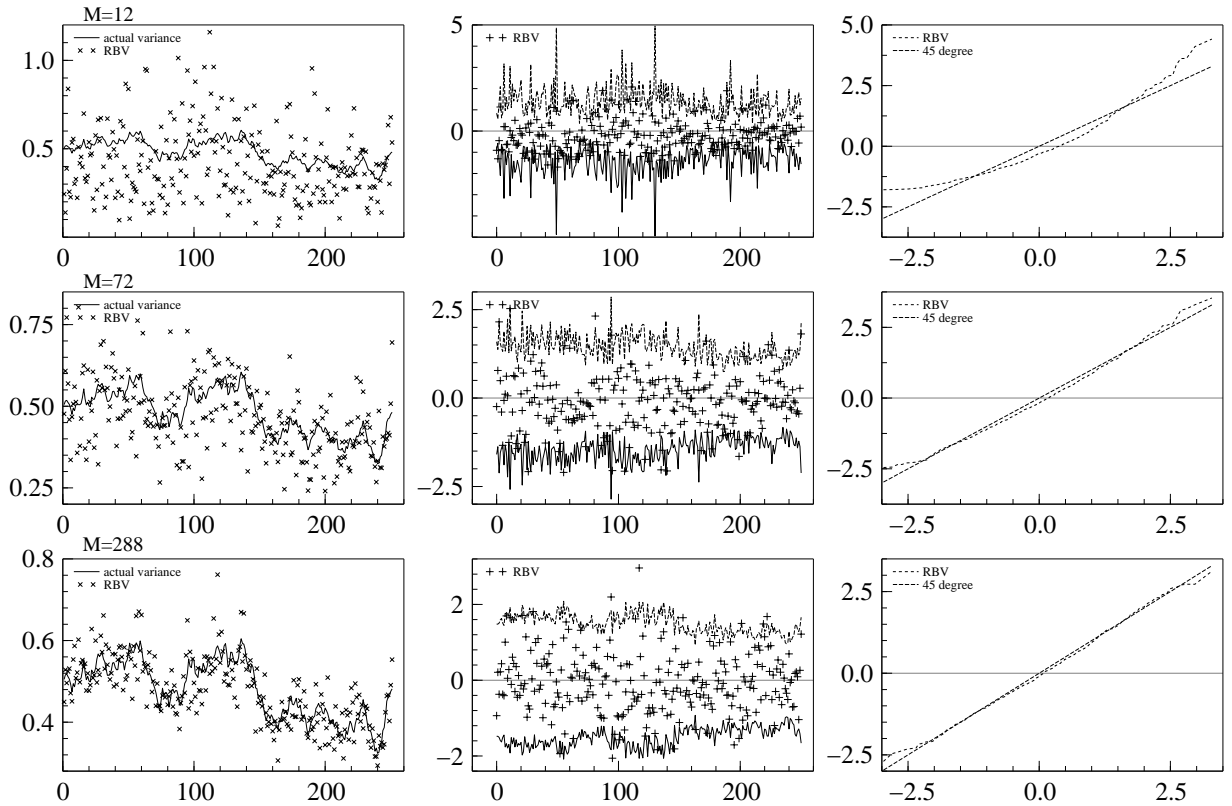


Figure 1: *Left: Actual variance and realised bipower variation. Middle: realised bipower variation minus actual variance and confidence interval. Right: QQ-plot of the realised bipower variation error.*

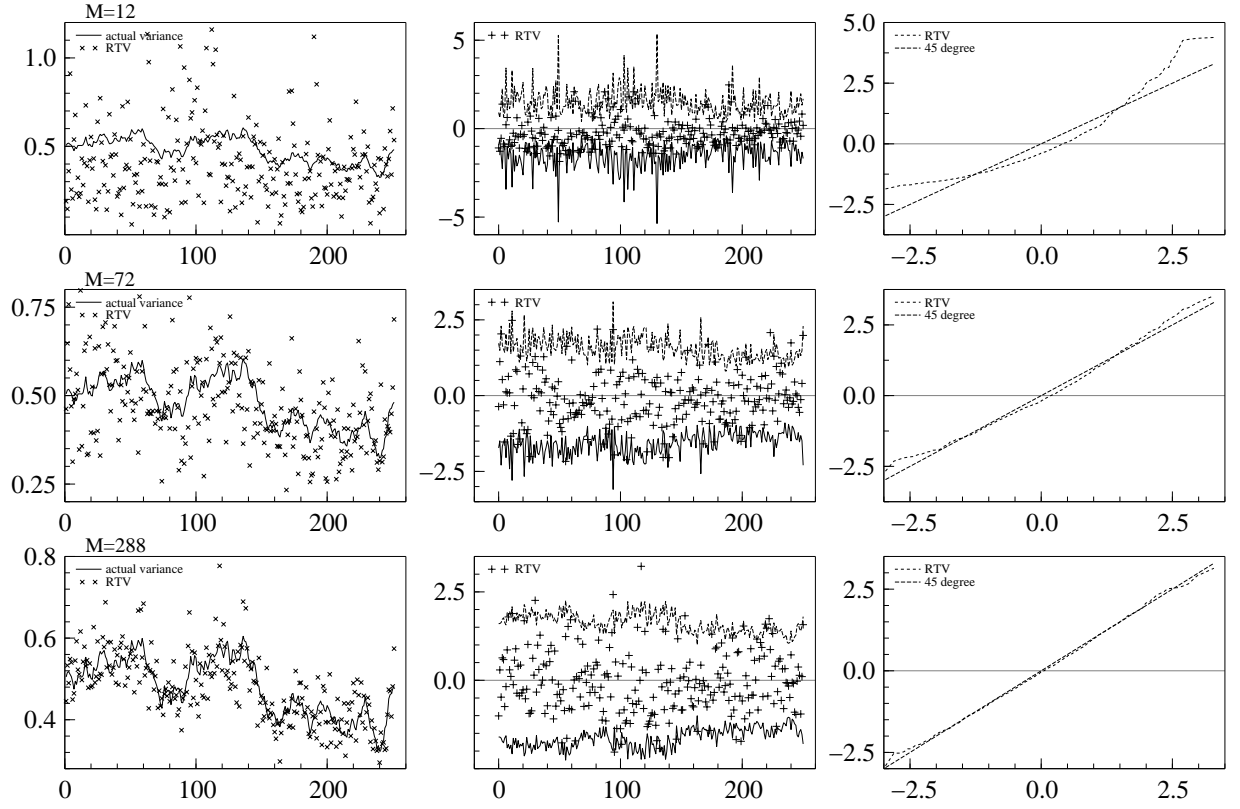


Figure 2: *Left: Actual variance and realised tripower variation. Middle: realised tripower variation minus actual variance and confidence interval. Right: QQ-plot of the realised tripower variation error.*

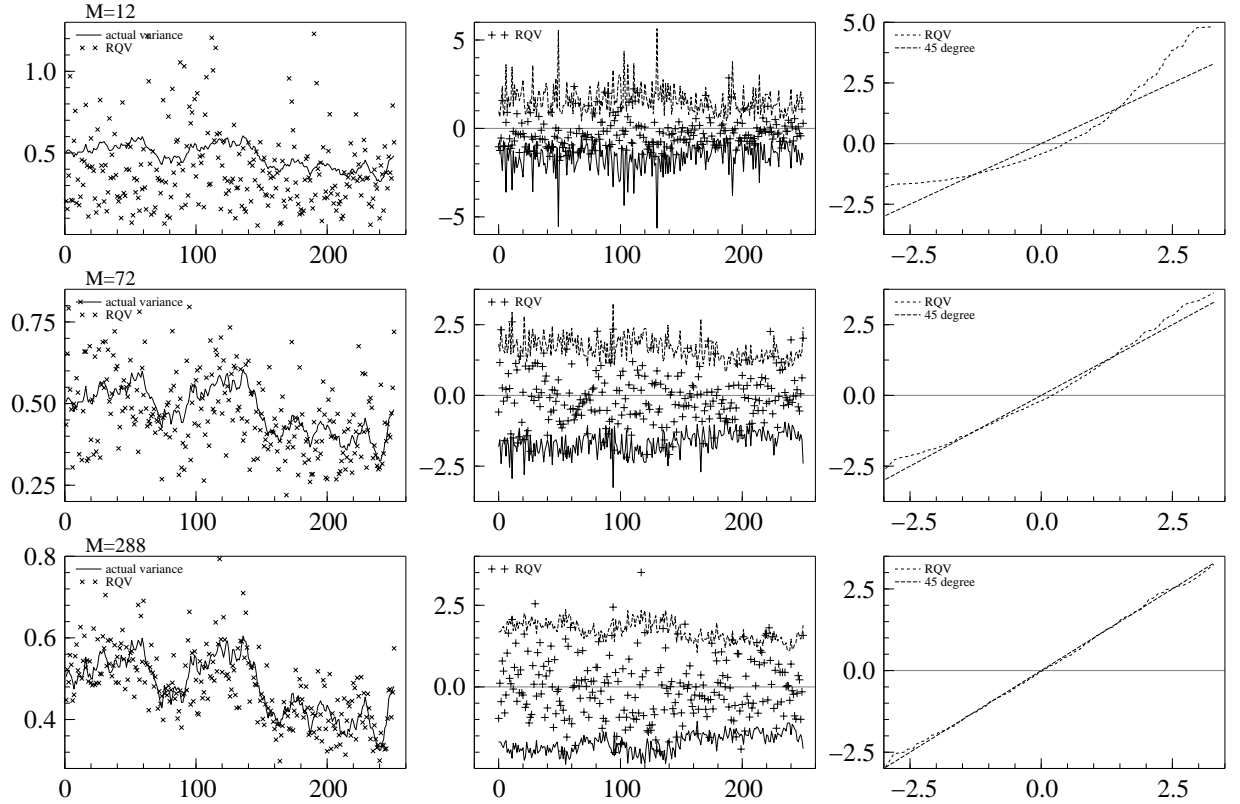


Figure 3: *Left: Actual variance and realised quadpower variation. Middle: realised quadpower variation minus actual variance and confidence interval. Right: QQ-plot of the realised quadpower variation error.*

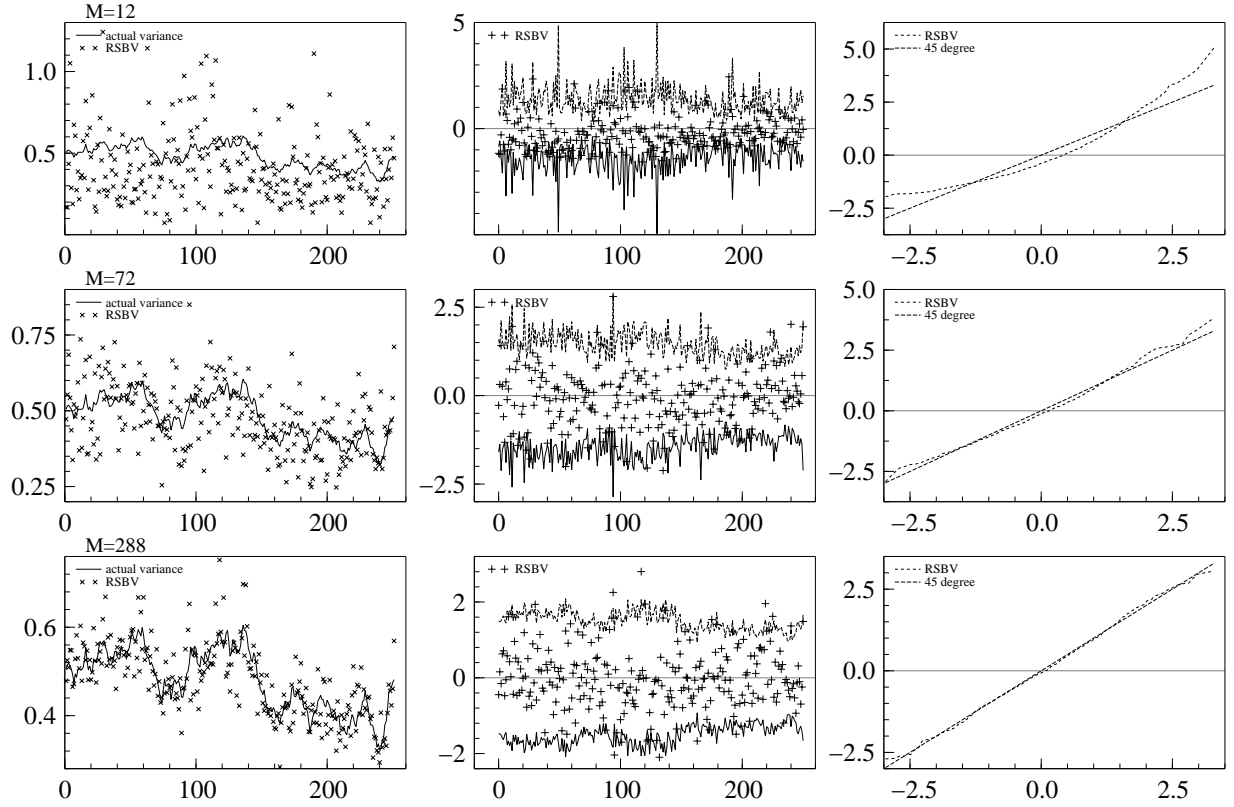


Figure 4: *Left: Actual variance and skipped version of realised bipower variation. Middle: skipped version of realised bipower variation minus actual variance and confidence interval. Right: QQ-plot of the skipped version of realised bipower variation error.*

An alternative view to assess the finite sample behaviour is given in Tables 1 and 2. They record the bias from zero and standard deviation of the errors (the corresponding realised series minus the actual variance). The bias should be zero and the standard deviation should be around one if the limit theory describes properly the behaviour of our statistics. The coverage rate is also reported. Given the asymptotic distributions of each of our estimators (see appendix), this rate is the percent of the standardised statistics which are under, in absolute values, the 97.5% quantile (setting the confidence level at 95%) of a Normal distribution. Table 1 gives the infeasible results (given that we are using simulations) and Table 2 gives the feasible results. In this last table we use the realised quarticity (E1) to estimate the integrated quarticity. As we are not including market microstructure noise, the results improve for larger values of  $M$ . All of the estimators seem to have a similar and good finite sample behaviour.

M	Bias	BV SD	Cov	Bias	TV SD	Cov	Bias	QV SD	Cov	Bias	SBV SD	Cov
12	-.189	.950	96.8	-.203	.992	95.6	-.208	1.02	95.4	-.216	.989	96.2
24	-.148	.971	96.1	-.146	.969	96.0	-.143	.971	96.1	-.138	.997	96.0
72	-.108	.995	95.2	-.085	.988	95.2	-.076	.993	95.5	-.070	1.01	94.8
144	-.074	1.01	94.6	-.081	1.02	94.6	-.089	1.01	94.8	-.071	1.01	95.1
288	-.047	.986	94.6	-.028	.990	95.2	-.021	.989	95.3	-.020	1.00	94.3
576	-.016	.983	96.2	-.023	.987	95.5	-.033	.986	95.2	-.026	1.00	95.4

Table 1: *Bias, standard deviation and coverage (95% level) of the infeasible standardised realised bipower, tripower, quad-power and skipped bipower variation error.*

M	Bias	BV SD	Cov	Bias	TV SD	Cov	Bias	QV SD	Cov	Bias	SBV SD	Cov
12	-.611	1.51	86.0	-.575	1.45	85.9	-.558	1.44	84.9	-.634	1.51	85.7
24	-.423	1.26	89.3	-.389	1.23	89.1	-.368	1.21	89.3	-.419	1.28	89.2
72	-.253	1.09	91.9	-.209	1.07	92.9	-.188	1.07	93.0	-.213	1.11	92.0
144	-.169	1.07	93.1	-.165	1.06	93.2	-.165	1.05	93.5	-.163	1.05	93.0
288	-.110	1.01	94.4	-.082	1.01	94.2	-.071	1.01	94.2	-.083	1.02	93.4
576	-.059	.994	95.9	-.061	.997	95.2	-.067	.996	95.0	-.071	1.01	95.3

Table 2: Bias, standard deviation and coverage (95% level) of the feasible standardised realised bipower, tripower, quad-power and skipped bipower variation error.

### 5.1.2. Finite sample behaviour in the presence of jumps

As stated previously, realised bipower variation and our three other estimators are robust to the presence of jumps. It will be convenient to study their finite sample behaviour in the presence of rare jumps (i.e. from a finite activity jump process). In Figure 5 we use RBV, RTV, RQV and RSBV to find the integrated variance and the quadratic variation of the jump component of the first 50 days of a series of simulated data. The data was simulated from the SV plus jump process explained previously. Realised values are calculated based on  $M=288$ . In the left hand side figures, the true integrated variance is shown together with the integrated variance estimated with the a)RBV, b)RTV, c)RQV and d)RSBV. For all the cases our estimators work rather well by not incorporating the variation due to the jumps. In the right hand side figures, the true quadratic variation of the jump component is shown together with the estimated one using a)RV-RBV, b)RV-RTV, c)RV-RQV and d)RV-RSBV. All of them seem to give accurate estimations but we need to apply the asymptotic distributions to obtain stronger conclusions.

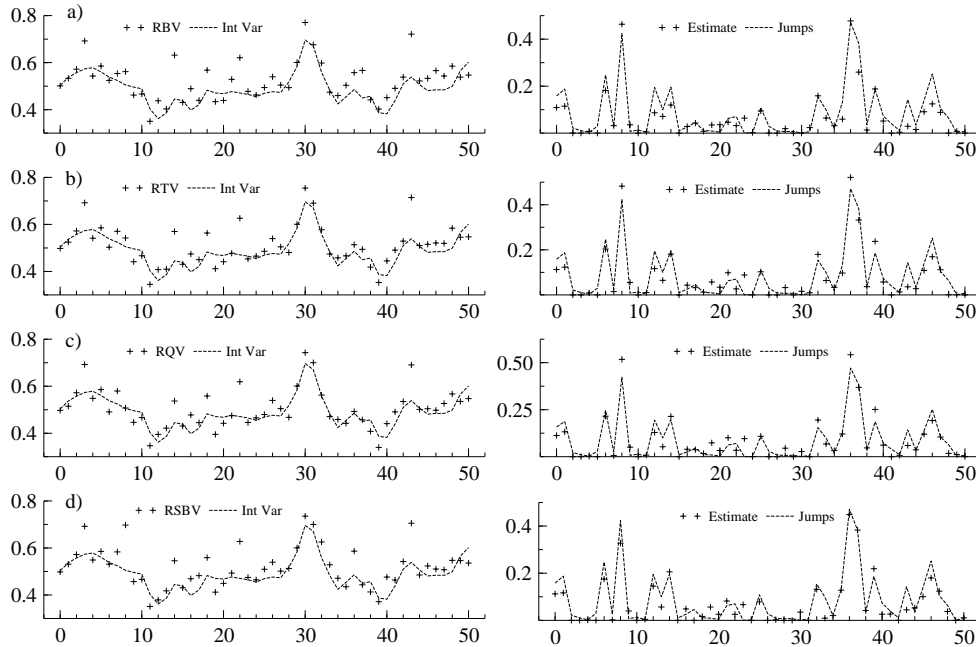


Figure 5: Simulation from a jump plus diffusion based SV model, estimating the integrated variance (left hand side figures) and the quadratic variation of the jump component (right hand side figures) with  $M=288$ . Rows correspond to RBV, RTV, RQV and RSBV respectively.



## 5.2. Null distribution

### 5.2.1. Infeasible test

Now instead of examining the finite sample behaviour of our estimators per se, we will investigate their accuracy in the tests for jumps. We will simulate a one thousand day process first without any jumps. We will do the infeasible linear and ratio tests for jumps using each one of the estimators (realised bipower variation, realised tripower variation, realised quadpower variation and realised skipped bipower variation) for different values of  $M$ , expecting not to reject the null hypothesis (the null hypothesis implies no jumps). The test statistics are plotted in Figure 6 for  $M=288$  for a)RBV, b)RTV, c)RQV and d)RSBV. In the left hand side graph the linear test statistics are shown together with twice their standard error. In the middle graph this is repeated for the ratio test. The hypothesis of no jumps will be rejected every time a value falls below twice<sup>3</sup> the standard error (setting a 95% confidence level), therefore the ratio test seems to give better results than the linear one. The right hand side graph shows the normal QQ-plot of the two tests. If the asymptotic approximations were accurate they should lay on the 45 degree line. Both test have good QQ-plots, although the one of the ratio test is slightly better for all our estimators.

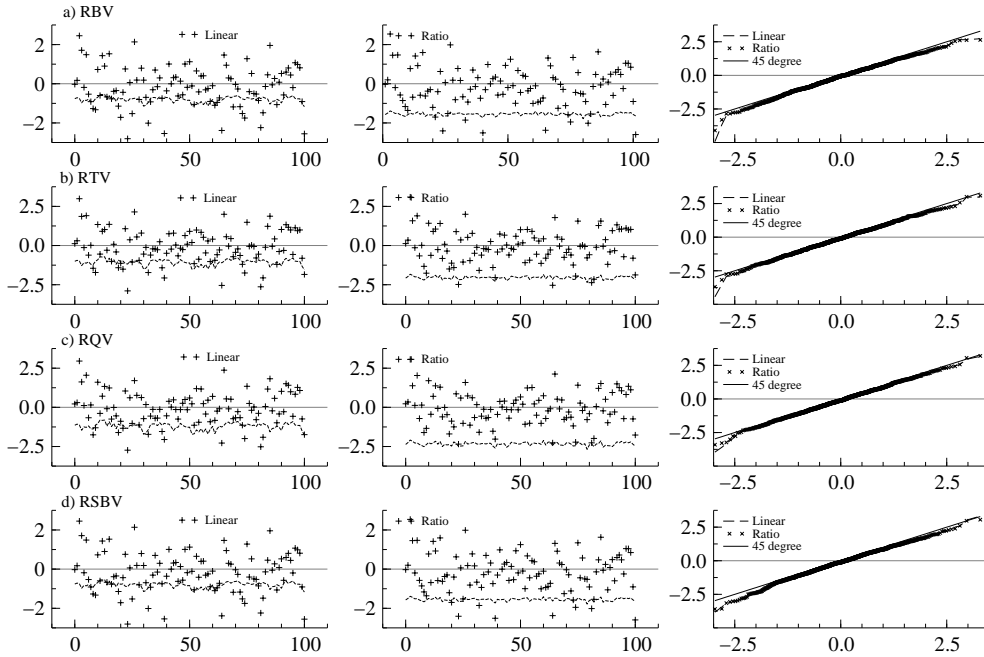


Figure 6: *T*-statistics plus twice their standard errors from simulations from the null distribution for the infeasible linear (left hand side) and ratio tests (middle) for  $M=288$  and for a)RBV, b)RTV, c)RQV and d)RSBV. QQ-plots are given in the right hand side.

In Table 3 the bias from zero of the linear test statistics, their standard deviation and the coverage rates for one-sided linear tests are recorded for the series using the four estimators (RBV, RTV, RQV, RSBV). The bias should tend to zero and the standard deviation to one if the asymptotic theory is describing properly the behaviour of the test statistic. The coverage rate tells the probability of not rejecting the null hypothesis when it is true, i.e., finding no jumps when the process is continuous. Here

<sup>3</sup>The 97.5% quantile of a standard Normal distribution is 1.96. We rounded it up and used two for the calculations.

we are taking  $\alpha = .05$  so the coverage rates should be  $(1 - \alpha) * 100\%$  if the asymptotic distributions fit properly the finite sample test statistics. In this table the infeasible test is done, i.e. the actual quarticity is used to calculate the linear test statistic.

In Table 4 the bias, standard deviation and coverage rate are reported for the infeasible ratio test. In both the linear and ratio tests we obtain better results as the value of  $M$  gets larger. In this case, the infeasible linear and ratio tests give very similar results. Realised tripower and quadpower variation give slightly better results than the estimators based on bipower variation. Realised bipower variation and its skipped version overestimate the jumps.

M	Bias	12 SD	Cove	Bias	72 SD	Cove	Bias	288 SD	Cove
RBV	-.363	1.16	87.4	-.191	1.00	92.8	-.115	1.00	93.5
RTV	-.313	1.13	88.5	-.180	1.00	92.9	-.103	.994	94.1
RQV	-.313	1.14	89.0	-.181	1.00	93.3	-.104	.999	94.3
RSBV	-.424	1.22	85.9	-.199	1.03	92.2	-.129	1.01	93.6

Table 3: *Bias, standard deviation and coverage (95% level) of the infeasible linear test.*

M	Bias	12 SD	Cove	Bias	72 SD	Cove	Bias	288 SD	Cove
RBV	-.379	1.08	85.2	-.196	.987	92.6	-.110	.993	93.3
RTV	-.328	1.07	89.5	-.185	.985	94.1	-.099	.985	93.7
RQV	-.329	1.08	89.7	-.183	.987	93.8	-.101	.992	94.4
RSBV	-.433	1.15	85.5	-.199	1.01	92.4	-.127	1.01	93.7

Table 4: *Bias, standard deviation and coverage (95% level) of the infeasible ratio test.*

### 5.2.2. Feasible tests

As explained previously the use of the integrated quarticity is infeasible in practice, so estimators for it should be used. The feasible test statistics, using the quadpower variation (E3) as the estimator of the integrated quarticity, are plotted now in Figure 7 for  $M=288$  for a)RBV, b)RTV, c)RQV and d)RSBV. The left hand side and middle graphs show the test statistics of the linear and ratio test respectively. The QQ-plots of both tests are shown in the right hand side graphs. Comparing this figure with Figure 6 (the feasible with the infeasible case), a deterioration is visible in the QQ-plots specially for the linear test.

In Table 5 the bias, standard deviation and coverage is reported for the linear test statistic using the realised quarticity (E1), the realised tripower variation with  $r = s = u = 4/3$  (E2) and the realised quadpower variation with  $r = s = u = v = 1$  (E3) given  $M=288$ . In Table 6 the results for the feasible ratio tests are shown for  $M=288$  using E1, E2 and E3 to estimate the actual quarticity. In the feasible case, confirming the results from the previous QQ-plots, the ratio tests give much better results than the linear tests. Realised tripower and quadpower variation outperform the other estimators when testing for jumps if the process is continuous. Bipower variation and its skipped version overestimate the presence of jumps.

In the case where the integrated quarticity is infeasible and there are no jumps in the process, the realised quarticity seems to be its best estimator (nevertheless when a jump component is present it

will be the worse one as explained later on). There is little to choose between its other two estimators, E2 and E3.

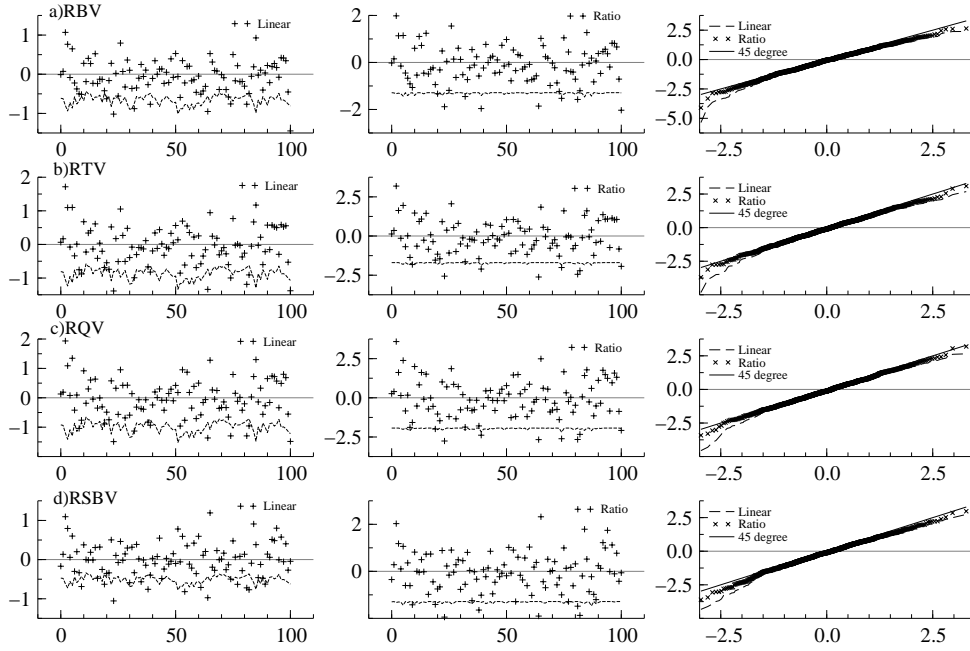


Figure 7: *T*-statistics plus twice their standard errors from simulations from the null distribution for the feasible linear (left hand side) and ratio tests (middle) for  $M=288$  and for a)RBV, b)RTV, c)RQV and d)RSBV. QQ-plots are given in the right hand side.

	Bias	E1 SD	Cove	Bias	E2 SD	Cove	Bias	E3 SD	Cove
RBV	-.086	.997	93.5	-.165	1.03	91.6	-.173	1.04	91.3
RTV	-.075	.992	94.3	-.154	1.02	92.1	-.167	1.03	91.6
RQV	-.076	.999	94.8	-.149	1.02	92.8	-.167	1.04	91.7
RSBV	-.103	1.01	94.4	-.161	1.04	92.9	-.175	1.05	92.2

Table 5: Bias, standard deviation and coverage (95% level) of the feasible linear test for  $M=288$ .

	Bias	E1 SD	Cove	Bias	E2 SD	Cove	Bias	E3 SD	Cove
RBV	-.041	.986	94.9	-.110	.980	93.7	-.113	.980	93.6
RTV	-.015	.987	96.4	-.094	.971	94.1	-.099	.976	94.1
RQV	-.007	.998	97.1	-.087	.968	94.8	-.097	.980	94.4
RSBV	-.056	.994	95.0	-.117	.985	94.1	-.124	.989	94.0

Table 6: Bias, standard deviation and coverage (95% level) of the feasible ratio test with  $M=288$ .

### 5.3. Alternative distribution

In this section we will add a jump component to the simulated process to assess the test for jumps when they are present in the log-price process. Each day the same number of jumps are added to the process (one thousand days will be simulated). The jumps will be i.i.d.  $N(0, \beta\nu/a)$  and several

simulations with a different  $\beta$  will be done to assess the performance of the test with different sizes of jumps. The one-sided hypothesis test is done every day and the level of significance is 5%. Table 7 reports the bias, standard deviation and the coverage for the linear infeasible test when one jump is added every day with  $\beta = 50\%$  and  $\beta = 20\%$ . In this case the coverage reports the percent of times we do an error type II, i.e. we accept the null hypothesis when there are jumps. It can be seen that as the size of the jump increases, the easier it is to detect the jump. As M becomes larger, the results of the hypothesis test improve. Almost no jumps are detected when the jumps are very small, especially when the sample frequency is low.

For the infeasible ratio test, the results for each one of the estimators (RV, RTV, RQV and RSBV) are reported in Table 8 when one jump is added each day. As under the null distribution, when we do the infeasible test there is not much difference between the linear and the ratio cases. Again if the size of the jump is small, it is more difficult to detect. As expected, as the value of M increases, the rate of accepting the null falls. Notice that the tests based on realised bipower variation and its skipped version seem to have more power than the ones based on the other estimators. This result can be misleading because the real size of the tests may be different for each estimator (in the previous section we found that the null distribution of the tests based on realised tripower and quadpower variation fit better the test statistics).

M	Bias	12 SD	Cove	Bias	72 SD	Cove	Bias	288 SD	Cove
50%									
RBV	−.728	1.75	78.1	−1.55	3.21	69.8	−3.49	6.42	55.0
RTV	−.639	1.60	79.5	−1.36	2.70	70.5	−2.91	5.19	56.5
RQV	−.623	1.55	79.7	−1.27	2.47	71.1	−2.65	4.66	57.7
RSBV	−.796	1.82	76.9	−1.54	3.05	69.2	−3.51	6.48	54.9
20%									
RBV	−.469	1.25	85.2	−.190	1.17	89.9	−.389	1.49	87.3
RTV	−.407	1.23	86.2	−.143	1.13	91.4	−.327	1.35	88.5
RQV	−.384	1.22	86.8	−.139	1.14	90.7	−.291	1.28	90.1
RSBV	−.505	1.28	83.9	−.224	1.15	90.6	−.399	1.46	87.0

Table 7: Bias, standard deviation and coverage (95% level) of the infeasible linear test when one jump is added every day.

M	Bias	12 SD	Cove	Bias	72 SD	Cove	Bias	288 SD	Cove
50%									
RBV	−.529	1.15	81.8	−.967	1.63	73.5	−2.18	2.99	57.0
RTV	−.467	1.11	84.3	−.862	1.44	75.2	−1.84	2.48	59.3
RQV	−.459	1.11	86.8	−.807	1.36	76.8	−1.68	2.26	60.5
RSBV	−.579	1.21	79.6	−.965	1.60	73.1	−2.19	3.03	57.8
20%									
RBV	−.421	1.06	86.7	−.172	1.08	90.8	−.331	1.24	88.2
RTV	−.361	1.04	88.4	−.129	1.06	92.4	−.278	1.15	90.0
RQV	−.347	1.04	89.9	−.123	1.07	92.0	−.246	1.12	91.0
RSBV	−.468	1.12	84.6	−.203	1.07	91.3	−.344	1.21	88.4

Table 8: Bias, standard deviation and coverage (95% level) of the infeasible ratio test when one jump is added every day.

Now we will do the feasible tests using the three estimators for the integrated quarticity ((E1), (E2) and (E3)). Table 9 shows the finite sample behaviour of the linear test when M=288. Immediately we can see how the realised quarticity is not giving adequate results when jumps are present in the process. Just as the realised variance does not give a good estimation of the integrated variance in

the presence of a jump component, the realised quarticity does poorly in estimating the integrated quarticity. The realised quarticity incorporates the variation due to the jump component and not just the one due to the continuous component. Nevertheless, the other two estimators, E2 and E3 give good and similar results.

Table 10 shows the finite sample behaviour for the feasible ratio test when  $M=288$ . When using the estimators E2 and E3 for the integrated quarticity, the results are encouraging and very similar to the ones obtained with infeasible tests. In contrast, when using the realised quarticity as the estimator, there is a little number of rejections of the null hypothesis.

	Bias	E1 SD	Cove	Bias	E2 SD	Cove	Bias	E3 SD	Cove
50%									
RBV	-.851	1.05	72.9	-3.22	5.72	55.4	-3.38	6.06	54.5
RTV	-.725	.961	84.6	-2.69	4.61	56.8	-2.83	4.91	56.6
RQV	-.669	.927	90.4	-2.44	4.13	58.4	-2.58	4.40	57.7
RSBV	-.843	1.05	73.1	-3.19	5.71	55.9	-3.37	6.09	55.3
20%									
RBV	-.218	1.03	91.5	-.427	1.45	86.0	-.436	1.47	85.7
RTV	-.179	.998	93.8	-.365	1.31	87.9	-.379	1.33	87.3
RQV	-.156	.988	94.8	-.323	1.25	88.9	-.343	1.27	88.5
RSBV	-.235	.995	92.5	-.415	1.40	86.2	-.433	1.42	86.2

Table 9: *Bias, st. deviation and coverage (95% level) of the feasible linear test when one jump is added every day for  $M=288$ .*

	Bias	E1 SD	Cove	Bias	E2 SD	Cove	Bias	E3 SD	Cove
50%									
RBV	-.667	.875	92.8	-2.10	2.88	58.2	-2.16	2.96	57.7
RTV	-.549	.808	96.6	-1.72	2.32	60.6	-1.79	2.42	59.9
RQV	-.498	.782	97.6	-1.54	2.07	62.4	-1.61	2.17	61.1
RSBV	-.656	.877	93.3	-2.07	2.87	59.2	-2.14	2.97	58.5
20%									
RBV	-.185	.956	94.2	-.328	1.22	88.4	-.331	1.23	88.3
RTV	-.143	.918	96.6	-.267	1.13	90.6	-.274	1.13	90.4
RQV	-.117	.906	97.3	-.228	1.08	91.9	-.239	1.02	91.3
RSBV	-.203	.924	95.8	-.327	1.16	88.7	-.337	1.18	88.5

Table 10: *Bias, standard deviation and coverage (95% level) of the feasible ratio test when one jump is added every day for  $M=288$ .*

#### 5.4. Size adjusted tests

When doing the previous tests for jumps, even if we set a confidence level of 95% the real size of the tests will not be exactly 5%. This is due to the fact that the asymptotic distributions do not fit properly our test statistics when using finite samples. When using bipower variation and its skipped version this problem is accentuated. Therefore it is inadequate to compare the results obtained from the tests where we added jumps to the log-price process. Each test has a different size depending on the estimator and the sample size used.

The real critical values corresponding to the desired confidence level need to be found. Given these critical values, size adjusted tests for jumps can then be done and compared.

Using Monte Carlo simulations, the real critical values for the tests for jumps with size of 5% are found for each test statistic and different values of  $M$ . In Table 11 the critical values (CV) of the linear infeasible tests are displayed with the quantiles of the Normal distribution (NQ) that correspond to them. The better the fit of the asymptotic distribution, the closer the critical value will be to the value of the 95<sup>th</sup> quantile. Now, using the simulated series where one daily jump was added to the log-price process the test for jumps can be repeated with a given fixed size of 5%. In Table 11 we give the power of the test (Pow) for different jump sizes ( $\beta = 50\%$  and  $20\%$ ), i.e. the percent of times we reject the null hypothesis given that there are jumps in the process. In Table 12 we display the results for the ratio infeasible tests.

From the tables we can confirm that the test statistics based in tripower and quadpower variation have better sample behaviour under the null distribution than the ones based on bipower variation and its skipped version. Also the ratio test statistics have better finite sample behaviour than the linear ones. Nevertheless it is not obvious how to conclude which of the tests have a better power. It seems that when the sample frequency is low, the tests based on tripower and quadpower variation have better power. Yet it seems this is not true for higher frequencies, the tests based on bipower variation and its skipped version seem to have a bigger power. Also in the case when the smaller jumps are added, tripower and quadpower variation outperform the bipower variation and the opposite happens when big jumps are added to the process.

M		12				72				288			
$\beta$	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	
RBV	2.46	99.3	0.092	0.063	1.93	97.3	0.199	0.057	1.85	96.8	0.343	0.085	
RTV	2.27	98.8	0.095	0.064	1.84	96.7	0.195	0.061	1.75	96.1	0.346	0.080	
RQV	2.09	98.2	0.106	0.067	1.80	96.4	0.184	0.069	1.71	95.7	0.337	0.077	
RSBV	2.63	99.6	0.093	0.056	1.88	97.1	0.202	0.062	1.83	96.7	0.353	0.076	

Table 11: Critical values (5% size), Normal quantiles and power of the infeasible linear tests when one jump is added every day.

M		12				72				288			
$\beta$	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	
RBV	2.24	98.8	0.064	0.050	1.85	96.9	0.170	0.054	1.82	96.6	0.331	0.076	
RTV	2.03	97.9	0.061	0.045	1.71	95.7	0.172	0.064	1.78	96.3	0.323	0.065	
RQV	1.99	97.7	0.045	0.039	1.76	96.1	0.147	0.067	1.72	95.8	0.309	0.062	
RSBV	2.36	98.1	0.068	0.042	1.84	96.8	0.174	0.061	1.79	96.4	0.336	0.073	

Table 12: Critical values (5% size), Normal quantiles and power of the infeasible ratio tests when one jump is added every day.

In Tables 13 and 14 the real critical values, the corresponding Normal quantiles and the power of the feasible linear and ratio tests are given using each of the estimators of the integrated quarticity (E1, E2, E3) when  $M=288$ . It can be seen immediately that the lowest power is obtained when using E1 as the estimator. The other two estimators give better and very similar results as expected. Also

there is not much to choose between the linear and ratio tests.

$\beta$	CV	E1 NQ	Pow 50%	Pow 20%	CV	E2 NQ	Pow 50%	Pow 20%	CV	E3 NQ	Pow 50%	Pow 20%
RBV	1.78	96.3	0.164	0.058	2.00	97.8	0.331	0.077	2.03	97.9	0.336	0.078
RTV	1.70	95.6	0.117	0.051	1.96	97.5	0.318	0.065	1.98	97.7	0.332	0.068
RQV	1.66	95.2	0.089	0.053	1.90	97.2	0.311	0.063	1.96	97.6	0.306	0.060
RSBV	1.77	96.2	0.167	0.048	1.96	97.5	0.333	0.068	1.98	97.7	0.338	0.070

Table 13: Critical values (5% size), Normal quantiles and power of the feasible linear tests for  $M=288$  when one jump is added every day.

$\beta$	CV	E1 NQ	Pow 50%	Pow 20%	CV	E2 NQ	Pow 50%	Pow 20%	CV	E3 NQ	Pow 50%	Pow 20%
RBV	1.63	94.9	0.078	0.052	1.79	96.4	0.330	0.078	1.80	96.5	0.331	0.077
RTV	1.52	93.6	0.070	0.042	1.73	95.9	0.315	0.064	1.75	96.1	0.317	0.062
RQV	1.46	92.8	0.062	0.051	1.65	95.2	0.310	0.062	1.70	95.6	0.307	0.063
RSBV	1.63	94.9	0.090	0.046	1.78	96.3	0.327	0.067	1.77	96.2	0.337	0.068

Table 14: Critical values (5% size), Normal quantiles and power of the feasible ratio tests for  $M=288$  when one jump is added every day.

From this section we can conclude that the realised tripower and quadpower variation give better results under the null hypothesis. Nevertheless once a jump component is added to the log-price process, the conclusion of which test gives the biggest power is unclear. Microstructure noise was not added to these simulations, so further research is needed to assess the power of test based on these estimators with contaminated series.

## 6. Empirical data

### 6.1. Testing for jumps

To illustrate empirically the test for jumps we will use the United States Dollar/ German Deutsche Mark series employed extensively in previous papers (see for example Andersen, Bollerslev, Diebold and Labys (2001), Barndorff-Nielsen and Shephard(2002)). This series covers from 1st of December 1986 until 30th of November 1996 and reports every five minutes the most recent quote on the Reuters screen. This dataset was kindly supplied by Olsen and Associates in Zurich (see Dacorogna, Gencay, Müller, Olsen and Pictet (2001)).

In this section we will study the feasible ratio test for jumps using the realised quadpower variation with  $r = s = u = v = 1$  (E3) to estimate the actual quarticity. In Figure 8 the ratio statistic

$$\delta^{-1/2} \left( \frac{\mu_r^{-n} \{Y_\delta\}_t^{[r_1, \dots, r_n]}}{[Y_\delta]_t^{[2]}} - 1 \right),$$

where  $r = r_1 = \dots = r_n$  and  $\sum_{i=1}^n r_i = 2$ , is plotted together with twice the standard error for the first 250 days for a)RBV, b)RTV, c)RQV and d)RSBV with  $M=144$ . In all of the cases, we can observe a considerable number of values below twice the standard error pointing out the existence of jumps in

the series. Just by looking at the graphs we cannot conclude anything about the different estimators so further results are reported in Table 15.

Table 15 shows for different values of  $M$  the sum of the first five correlation coefficients of the Dollar/DM series and the average values over the sample of the realised variance and of the bipower, tripower, quadpower and skipped version of bipower variation. It also shows for each of these estimators the proportions of rejections of the null hypothesis at the 5% level and 1% level.

Firstly we can see how the sum of correlations increases when the value of  $M$  increases, reaching large values for  $M=144$  (-.056) and  $M=288$  (-.092) possibly explained by the microstructure effect. This fact surely is affecting the number of rejections of the null hypothesis as it decreases when  $M=288$  for most of the estimators.

From the simulations in the previous section we know that the tests based on realised tripower and quadpower variation have bigger sizes (given a confidence level) compared to the tests based on realised bipower variation when using finite samples. This may explain the values in Table 15 for  $M=12, 24$  and  $72$ . This fact reverses when  $M=144$  and  $288$  perhaps due to the microstructure effect, this gives us a first reason to think realised tripower and quadpower variation are more robust in the presence of such microstructure noise. The bias of the skipped version of realised bipower variation is almost the same as the one of realised tripower variation, nevertheless the proportion of rejections of the null hypothesis is higher. In fact, it is the highest compared to all the other estimators and it is the only estimator for which the number of rejections are not reduced when  $M=288$ .

Although these give us a preliminary insight into the behaviour of these estimators in the presence of microstructure noise, further research has to be done to obtain stronger conclusions. Market microstructure noise may hide the jump component of a process so we can expect that under this noise our tests will underestimate the presence of jumps. Consequently, what we can conclude now is that certainly the Dollar/DM series contains jumps.

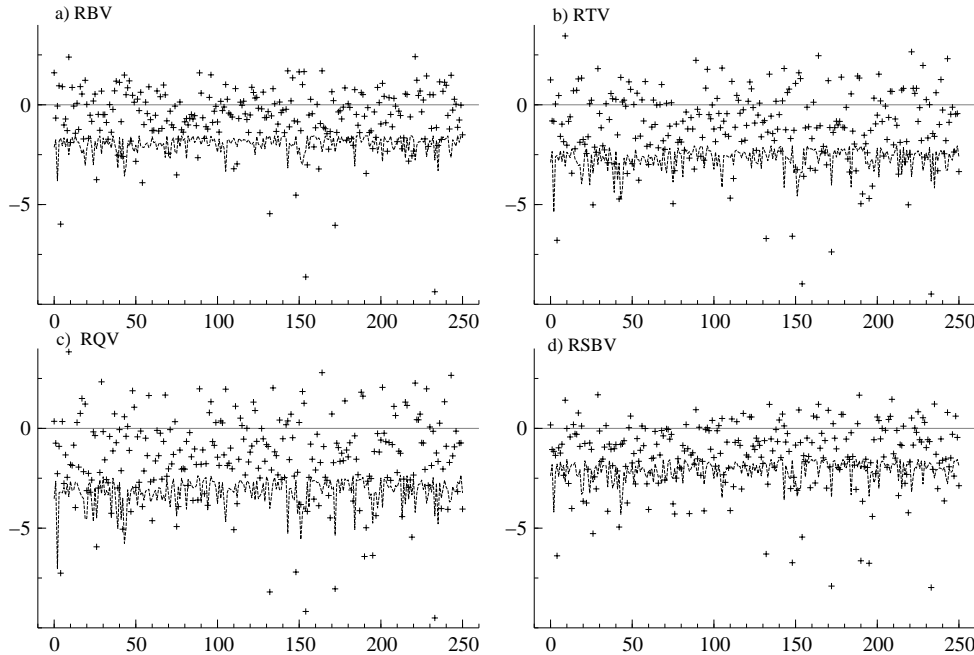


Figure 8: Ratio  $t$ -statistics plus twice their standard errors from the first 250 observations of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with  $M=144$ .



M	corr	RV	BV	5%r	1%r	TV	5%r	1%r	QV	5%r	1%r	SB	5%r	1%r
12	.001	.459	.387	.183	.081	.357	.164	.043	.339	.140	.024	.357	.218	.098
24	.002	.463	.396	.207	.101	.372	.192	.078	.354	.177	.051	.373	.232	.118
72	-.001	.488	.442	.224	.120	.416	.233	.109	.402	.217	.089	.415	.283	.157
144	-.056	.511	.473	.228	.116	.448	.250	.126	.432	.255	.113	.442	.341	.186
288	-.092	.532	.502	.188	.095	.481	.227	.108	.467	.252	.107	.471	.368	.211

Table 15: Sum of the first five correlation coefficients of the Dollar/ DM series (*corr*). Average value of realised variance (*RV*), realised bipower (*BV*), tripower (*TV*), quadpower (*QV*) and skipped bipower (*SB*) variation. Proportion of rejections of the null hypothesis at the 5% (5%r) and 1% (1%r) level.

## 6.2. Case study

As explained before a large value of the realised variance can be explained by the presence of jumps or by high volatility in the continuous component. As in Barndorff-Nielsen and Shephard (2006) we will contrast two days, one in which the extreme value of the quadratic variation is caused by the jump component and in another one where it is caused by the continuous component. We will compare the behaviour of each of our estimators of integrated variance on these days.

On one of the days, January 15th 1988, we have a huge increase in the Dollar/DM rate (Figure 9, upper graph). This increase is reflected in the realised variance and not in the other estimators of the integrated variance indicating the presence of a jump component. In contrast on August 19th 1991 (Figure 9, lower graph) there is a constant increase of the rate. Here the large estimated value of the quadratic variation is due to the integrated variance so we cannot expect it to be caused by a jump but by high volatility.

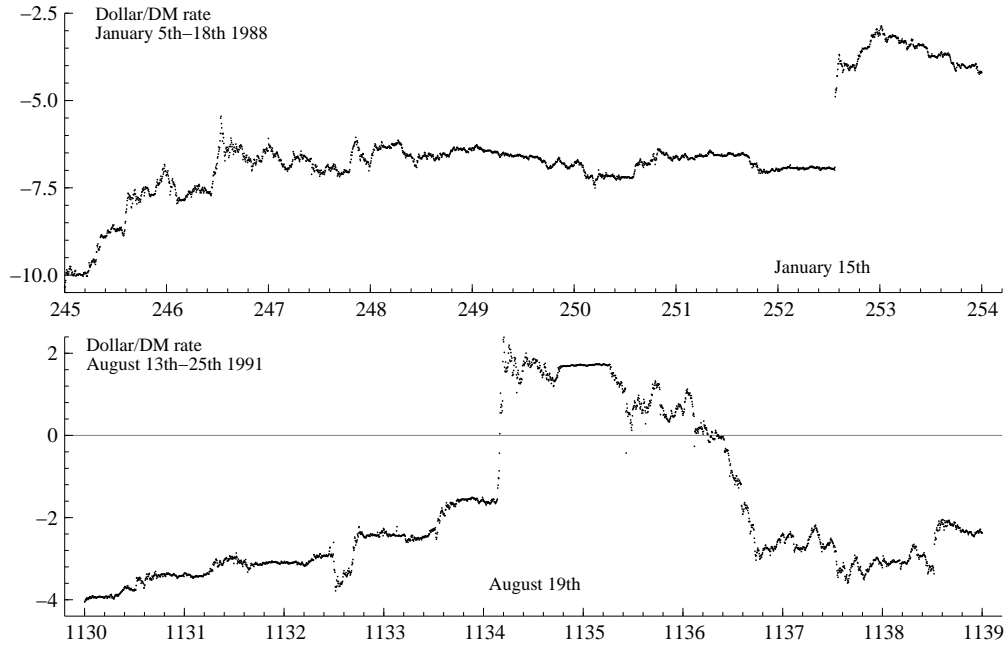


Figure 9: Dollar/DM rates from January 5th to January 18th 1988 (upper graph) and from August 13th to August 25th 1991 (lower graph). In the upper graph the large realised variance is due to a jump component and in the lower graph it is due to high volatility in the continuous component.

In Figure 10 we look carefully at the ratio  $t$ -statistics of the days around January 15th 1988. In all the cases ( a)RBV, b)RTQ, c)RQV and d)RSBV ) there is no doubt about the presence of a jump, the statistics are significantly below the 99% critical values. We reject the null hypothesis of no jumps independently of the estimator used.

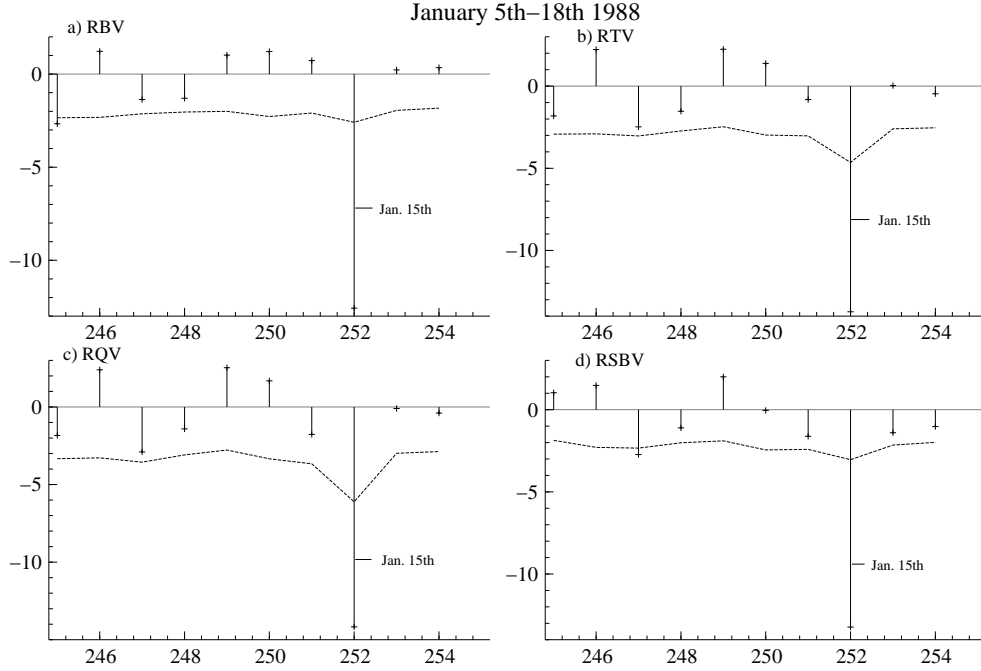


Figure 10: *January 5th to January 18th 1988. Ratio  $t$ -statistics and 99% critical values of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with  $M=288$ . On January 15th a jump occurred.*

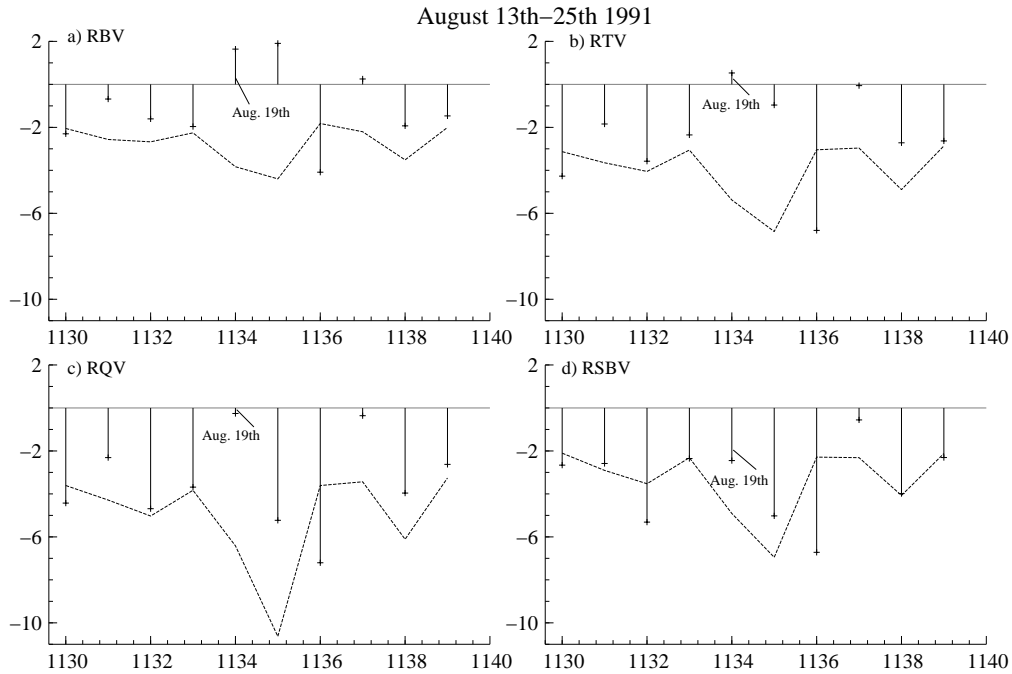


Figure 11: *August 13th to August 25th 1991. Ratio  $t$ -statistics and 99% critical values of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with  $M=288$ . On August 19th the large value of the realised variance is caused by high volatility on the continuous component and not a jump.*

In Figure 11 we plot again the ratio t-statistic along with the 99% critical values using our four estimators but now for the days around August 19th 1991. In all the cases we reject the presence of jumps, instead the increase of the quadratic variation is due to a quick movement in the rates.

Note that all the estimators give the same result for August 19th but not for the surrounding days. When using the skipped version of the realised bipower variation three jumps are found during the period studied.

## 7. Conclusions

A commonly used and well studied estimator of integrated variance is the realised variance. Nevertheless, the fact that absolute returns are more robust than squares when estimating the variability of returns forces us to look at the generalisation of quadratic variation called multipower variation. This measure, specifically bipower variation, was originally proved to be robust to jumps in Barndorff-Nielsen and Shephard (2004a). In the case of a stochastic volatility plus infrequent jump process, the quadratic variation of the jump component can be estimated by the difference between the realised variance and realised bipower variation. This fact provides a tool to test for the presence of jumps in the log-price process (as in Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2006), Huang and Tauchen (2005), Tauchen and Zhou (2006)). Alternative estimators based on multipower variation are also robust to jumps, i.e. tripower and quadpower variation.

The aim of this paper was to study these alternative estimators to test for the presence of jumps in the log-price process. For that we provided the necessary asymptotic distributions (for the differences between realised variance and realised tripower variation, realised variance and realised quadpower variation and realised variance and realised skipped bipower variation) and developed linear and ratio tests for jumps. To our knowledge this is the first time tests for jumps have been implemented using tripower, quadpower and the skipped version of bipower variation.

Using simulated data we could determine that realised tripower and quadpower variation are reliable, giving even better results than the realised bipower variation under the null hypothesis. The test statistics based on realised tripower and quadpower variation seem to have better finite sample behaviour, nevertheless it is unclear which of the tests is the most powerful. As expected, the performance of all the tests improves when using large values of  $M$ .

Once jumps were added to the process the power of the tests depended on the sample frequency and the size of the jumps, so it is not possible to give a conclusion about which of the estimators gave a better power. The addition of finite or infinite activity jump processes and the magnitude of the jumps will obviously determine the power of our tests and the robustness of our asymptotic theories. This issue needs further attention and has recently been examined in Barndorff-Nielsen, Shephard and Winkel (2005).

When using a feasible test, the integrated quarticity needs to be estimated. Under the presence of jumps, realised quarticity gives a great number of rejections of the null hypothesis as it estimates poorly the integrated quarticity. The other estimators give encouraging results.

The tests for jumps were also applied to empirical data, the Dollar/DM series. Results need to be interpreted carefully when working with real data, as it is well known that the price process is contaminated by market microstructure effects. Even though the previous tests for jumps were based

on asymptotic results, its implementation using extremely large values of  $M$  can give misleading answers as this noise accumulates. The larger the value of  $M$  we use, the closer we get to the asymptotic results but also the more the microstructure effect will disturb the real process.

By using the skipped version of the realised bipower variation, i.e. realised skipped bipower variation and by giving less weight to adjacent observations, i.e. realised tripower and quadpower variation, we may lessen this problem but further work is needed to assess the effectiveness of these estimators in the presence of microstructure noise.

## 8. Appendix: Proofs

For the following three proofs let us first define the generalised multipower variation as

$$Y^M(g)_t = \frac{1}{M} \sum_{i=1}^{\lfloor Mt \rfloor} \left( \prod_{i'=1}^{I \wedge (i+1)} g_{i'}(\sqrt{M} y_{i+i'-1}) \right).$$

As in Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndoff-Nielsen, Graversen, Jacod and Shephard (2006) let us assume that all the  $g_i$  are continuous with at the most polynomial growth and that  $Y \in SVSM^c$ . So if  $g_i(y) = |y|^{2/I}$ , we get

$$Y(g)_t = \mu_{2/I}^I \int_0^t \sigma_s^2 ds,$$

Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) show that

$$\sqrt{M}(Y^M(g)_t - Y(g)_t) \rightarrow \int_0^t \sqrt{\omega_I^2 \sigma_s^4} dB_s \quad (2)$$

where

$$\omega_I^2 = Var\left(\prod_{i=1}^I |u_i|^{2/I}\right) + 2 \sum_{j=1}^{I-1} Cov\left(\prod_{i=1}^I |u_i|^{2/I}, \prod_{i=1}^I |u_{i+j}|^{2/I}\right).$$

### A.1 Tripower Variation

We are mainly interested in the distribution of the following object

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]}.$$

To obtain it, we need to find the joint distribution of the realised variance error and the realised tripower variation error, i.e., the distribution of

$$\begin{pmatrix} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{2/3}^{-3} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \int_0^t \sigma_s^2 ds \end{pmatrix}.$$

By Barndorff-Nielsen and Shephard (2002) we know that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left( \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, 2).$$

A particular case of equation (2) gives us

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left( \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \mu_{2/3}^3 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_3^2)$$

where

$$\begin{aligned} \omega_3^2 &= \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ &\quad + 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) \\ &\quad + 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}) \end{aligned}$$

where  $u, u', u''$  and  $u'''$  are iid Normal(0,1).

Remember that  $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$ , so if  $A_t = 0$  we obtain that

$$\begin{aligned} \delta^{-1} \text{Cov} \left( \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \begin{pmatrix} \nu_j^2 (u_j^2 - 1) \\ \nu_j^{2/3} \nu_{j+1}^{2/3} \nu_{j+2}^{2/3} (|u_j|^{2/3} |u_{j+1}|^{2/3} |u_{j+2}|^{2/3} - \mu_{2/3}^3) \end{pmatrix} \right) \\ \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds \end{aligned}$$

where

$$\begin{aligned} k_1 &= \text{Var}(u^2) \\ k_2 &= k_3 = 3\text{Cov}(u^2, |u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ k_4 &= \omega_3^2 = \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ &\quad + 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) \\ &\quad + 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}). \end{aligned}$$

This implies that

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \begin{pmatrix} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{2/3}^{-3} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \int_0^t \sigma_s^2 ds \end{pmatrix}$$

$$\begin{aligned} &\xrightarrow{L} N\left(0, \begin{pmatrix} k_1 & \mu_{2/3}^{-3} k_2 \\ \mu_{2/3}^{-3} k_3 & \mu_{2/3}^{-6} k_4 \end{pmatrix}\right) \\ &\simeq N\left(0, \begin{pmatrix} 2 & 2 \\ 2 & 3.0613 \end{pmatrix}\right) \end{aligned}$$

by using the results that

$$\begin{aligned} \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) &= \mu_{4/3}^3 - \mu_{2/3}^6 \\ \text{Cov}(u^2, |u|^{2/3} |u'|^{2/3} |u''|^{2/3}) &= \mu_{8/3} \mu_{2/3}^2 - \mu_{2/3}^3 \\ \text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) &= \mu_{4/3}^2 \mu_{2/3}^2 - \mu_{2/3}^6 \\ \text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}) &= \mu_{4/3} \mu_{2/3}^4 - \mu_{2/3}^6 \end{aligned}$$

where  $\mu_r = E(|x|^r)$ .

With this joint distribution, the limit theory necessary for the test for jumps can be obtained

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( [Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \right) \xrightarrow{L} N(0, \vartheta_{TV})$$

where

$$\vartheta_{TV} = \mu_{4/3} \mu_{2/3}^{-2} (\mu_{4/3}^2 \mu_{2/3}^{-4} + 2\mu_{4/3} \mu_{2/3}^{-2} - 2) - 7 \simeq 1.0613.$$

## A.2 Quadpower Variation

Now we are interested in the distribution of the following object

$$[Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]}.$$

We need to find the following joint distribution

$$\left( \begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|y_j| |y_{j+1}| |y_{j+2}| |y_{j+3}|} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \end{array} \right)$$

From equation (2) we can obtain that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left( \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^{1/2} |y_{j+1}|^{1/2} |y_{j+2}|^{1/2} |y_{j+3}|^{1/2} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_4^2)$$

where

$$\begin{aligned} \omega_4^2 &= \text{Var}(\sqrt{|u| |u'| |u''| |u'''|}) \\ &\quad + 2\text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u'| |u''| |u'''| |u^{(IV)}|}) \end{aligned}$$

$$\begin{aligned}
& +2Cov(\sqrt{|u||u'||u''||u'''|}, \sqrt{|u''||u'''||u^{(IV)}||u^{(V)}|}) \\
& +2Cov(\sqrt{|u||u'||u''||u'''|}, \sqrt{|u'''||u^{(IV)}||u^{(V)}||u^{(VI)}|}).
\end{aligned}$$

If  $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$  and  $A_t = 0$  then

$$\begin{aligned}
& \delta^{-1}Cov\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \left( \frac{\nu_j^2(u_j^2 - 1)}{\sqrt{\nu_j\nu_{j+1}\nu_{j+2}\nu_{j+3}}}(\sqrt{|u_j||u_{j+1}||u_{j+2}||u_{j+3}|} - \mu_{1/2}^4) \right) \right. \\
& \quad \left. \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds \right)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= Var(u^2) \\
k_2 &= k_3 = 4Cov(u^2, \sqrt{|u||u'||u''||u'''|}) \\
k_4 &= \omega_4^2 = Var(\sqrt{|u||u'||u''||u'''|}) \\
& +2Cov(\sqrt{|u||u'||u''||u'''|}, \sqrt{|u'||u''||u'''||u^{(IV)}|}) \\
& +2Cov(\sqrt{|u||u'||u''||u'''|}, \sqrt{|u''||u'''||u^{(IV)}||u^{(V)}|}) \\
& +2Cov(\sqrt{|u||u'||u''||u'''|}, \sqrt{|u'''||u^{(IV)}||u^{(V)}||u^{(VI)}|}).
\end{aligned}$$

Implying that

$$\begin{aligned}
& \frac{1}{\delta^{1/2}\sqrt{\int_0^t \sigma_s^4 ds}} \left( \begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{1/2}^{-4} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|y_j||y_{j-1}||y_{j-2}||y_{j-3}|} - \int_0^t \sigma_s^2 ds \end{array} \right) \\
& \xrightarrow{L} N \left( 0, \begin{pmatrix} k_1 & \mu_{1/2}^{-4}k_2 \\ \mu_{1/2}^{-4}k_3 & \mu_{1/2}^{-8}k_4 \end{pmatrix} \right) \\
& \simeq N \left( 0, \begin{pmatrix} 2 & 2 \\ 2 & 3.37702 \end{pmatrix} \right)
\end{aligned}$$

as

$$\begin{aligned}
Var(|u|^{1/2}|u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}) &= \mu_1^4 - \mu_{1/2}^8 \\
Cov(u^2, |u|^{1/2}|u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}) &= \mu_{5/2}\mu_{1/2}^3 - \mu_{1/2}^4 \\
Cov(|u|^{1/2}|u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}, |u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}|u^{(IV)}|^{1/2}) &= \mu_1^3\mu_{1/2}^2 - \mu_{1/2}^8 \\
Cov(|u|^{1/2}|u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}, |u''|^{1/2}|u'''|^{1/2}|u^{(IV)}|^{1/2}|u^{(V)}|^{1/2}) &= \mu_1^2\mu_{1/2}^4 - \mu_{1/2}^8 \\
Cov(|u|^{1/2}|u'|^{1/2}|u''|^{1/2}|u'''|^{1/2}, |u'''|^{1/2}|u^{(IV)}|^{1/2}|u^{(V)}|^{1/2}|u^{(VI)}|^{1/2}) &= \mu_1\mu_{1/2}^6 - \mu_{1/2}^8.
\end{aligned}$$

Finally we can obtain the limit distribution that we were originally looking for

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( [Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \right) \xrightarrow{L} N(0, \vartheta_{QV})$$

where

$$\vartheta_{QV} = \mu_1 \mu_{1/2}^{-2} (\mu_1^3 \mu_{1/2}^{-6} + 2\mu_1^2 \mu_{1/2}^{-4} + 2\mu_1 \mu_{1/2}^{-2} - 2) - 9 \simeq 1.37702$$

and  $\mu_r = E(|x|^r)$ .

### A.3 Skipped Bipower Variation

Lastly we will obtain the asymptotic distribution of the difference of the realised variance errors and the skipped version of the realised bipower variation errors,

$$[Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1, 0, 1]}.$$

The joint distribution we need to find is

$$\left( \begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \mu_1^2 \int_0^t \sigma_s^2 ds \end{array} \right).$$

From equation (2) we obtain that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left( \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \mu_1^2 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_{2SV}^2)$$

where

$$\omega_{2SV}^2 = Var(|u| |u''|) + 2Cov(|u| |u''|, |u''| |u^{(IV)}|).$$

Note that the term  $Cov(|u| |u''|, |u'| |u'''|)$  is not included as it equals zero.

Given that  $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$  and  $A_t = 0$ , now we can obtain that

$$\begin{aligned} \delta^{-1} Cov \left( \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \left( \frac{\nu_j^2 (u_j^2 - 1)}{\sqrt{\nu_j^2} \sqrt{\nu_{j+2}^2} (|u_j| |u_{j+2}| - \mu_1^2)} \right) \right) \\ \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds \end{aligned}$$

where

$$\begin{aligned} k_1 &= Var(u^2) \\ k_2 &= k_3 = 2Cov(u^2, |u| |u''|) \\ k_4 &= \omega_{2SV}^2 = Var(|u| |u''|) + 2Cov(|u| |u''|, |u''| |u^{(IV)}|). \end{aligned}$$



The joint distribution can now be expressed as

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( \begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_1^{-2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \int_0^t \sigma_s^2 ds \end{array} \right) \xrightarrow{L} N \left( 0, \begin{pmatrix} k_1 & \mu_1^{-2} k_2 \\ \mu_1^{-2} k_3 & \mu_1^{-4} k_4 \end{pmatrix} \right) \\ \simeq N \left( 0, \begin{pmatrix} 2 & 2 \\ 2 & 2.60907 \end{pmatrix} \right)$$

as

$$\begin{aligned} \text{Var}(u^2) &= 2 \\ \text{Var}(|u| |u''|) &= 1 - \mu_1^4 \\ \text{Cov}(|u| |u''|, |u''| |u^{(IV)}|) &= \mu_1^2 (1 - \mu_1^2) \\ \text{Cov}(u^2, |u| |u''|) &= \mu_3 \mu_1 - \mu_1^2. \end{aligned}$$

Therefore the limit theory is

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left( [Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,0,1]} \right) \xrightarrow{L} N(0, \vartheta_{SBV})$$

where

$$\vartheta_{SBV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.60907$$

and  $\mu_r = E(|x|^r)$ .

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